Interaction between a Line Soliton and a *Y*-Periodic Soliton in the $(2+1)$-dimensional Nizhnik-Novikov-Veselov Equation

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A variable separation approach is used to obtain exact solutions of the Nizhnik-Novikov-Veselov equation. Some exact solutions of this model are analysed to study the interaction between a line soliton and a *y*-periodic soliton. The interactions are classified into several types according to the phase shifts due to the collision. There are two types of singular interactions: One is the resonant interaction that generates one line soliton, while the other is the extremely repulsive or long-range interaction where two solitons interchange infinitely apart. Detailed behaviors of interactions are illustrated both analytically and graphically.

Key words: Interaction; Variable Separation Approach; NNV Equation.

1. Introduction

$(1+1)$-dimensional solitary wave solutions have been widely applied in fluid mechanics, plasma physics, optics, etc. [1]. In $(2+1)$-dimensions some significant integrable equations such as the Kadomtsev-Petviashvili (KP) equation [2], the Davey-Stewartson (DS) equation [3], the Nizhnik-Novikov-Veselov (NNV) [4] equation, the breaking soliton equation [5], etc., have been established in nonlinear physics.

The interaction of solitons plays a role in many applications, and therefore, its study for integrable models is important.

From a symmetry study we know that there exist quite rich symmetry structures with arbitrary functions for $(2+1)$-dimensional integrable models [6]. Therefore the soliton structure and the interaction between solitons of $(2+1)$-dimensional nonlinear models may show quite rich phenomena that have not yet been revealed.

Because the finding of physically significant soliton solutions in $(2+1)$-dimensions is much more difficult than in $(1+1)$-dimensions, the study of the interaction between solitons of $(2+1)$-dimensional nonlinear models is very difficulty. Recently, two types of variable-separating procedures have been established for the nonlinear case. The second type will be used here to obtain some exact solutions of the NNV equation. Using these exact solutions, we will explore the interaction properties between such solitons. The interaction between a line soliton and a *y*-periodic soliton will be studied in this paper.

The paper is organized as follows: In Sect. 2 we use the second type of variable separation to find some special solutions of the NNV equation. The interaction between a line soliton and a *y*-periodic soliton is discussed in Sect. 3, while a summary and a discussion follow in Section 4.

2. Exact Solutions of the $(2+1)$-dimensional NVV Equation

The $(2+1)$-dimensional NVV equation

$$u_t + u_{xxx} + u_{yyy} = 3(uv)_x + 3(uw)_y, \quad (1)$$

$$v_y = u_x \quad , \quad w_x = u_y. \quad (2)$$

is an isotropic Lax integrale extension of the well known $(1+1)$-dimensional KdV equation. Soliton solutions of the NNV equation have been studied by many authors. For instance, Boiti et al. [7] solved the NNV equation via the inverse scattering transformation. Radha and Lakshmanan obtained the multi-dromion solutions by means of the bilinear method

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[8]. Tagami, Hu, and Li [9] obtained soliton-like solution of the NNV equation by means of the Bäcklund transformation. However, the NNV equation yields many interesting soliton structures that have not yet been found, and the interaction between the solitons is still not clear. By the variable separation method we can reveal new phenomena of interaction.

To use the variable separation approach, we perform Bäcklund transformation:

$$u=-2(\ln f)_{xy}+u_0,\quad v=-2(\ln f)_{xx}+v_0,\quad w=-2(\ln f)_{yy}+w_0, \tag{3}$$

where $\{u_0, v_0, w_0\}$ is an arbitrary known seed solution of the NNV equation. For simplicity, we consider the special case

$$u_0=0,\quad v_0=v_0(x,t),\quad w_0=w_0(y,t). \tag{4}$$

Substituting (3) with (4) into (1) leads to the trilinear form

$$\begin{aligned}&-f(f_{xxxx}f_y+f_xf_{yt}-2f_{xy}f_{yyy}+f_tf_{xy}+f_xf_{yyyy}+4f_{xxxy}f_x)\\&+2f_{xxx}(ff_{xy}+f_xf_y)+6f_x^2f_{xxy}-6f_{xy}f_xf_{xx}+6f_y^2f_{yyx})\\&-f_y(4ff_{yyyx}+ff_{xt}-2f_xf_t-2f_xf_{yyy}+6f_{xy}f_{yy})\\&+f^2(f_{xxxxy}+f_{yyyyx}+f_{xyt})+3f(f_xf_y-ff_{xy})(w_{0y}+v_{0x})\\&+v_0(6ff_xf_{xy}+3ff_yf_{xx}-3f^2f_{xxy}-6f_x^2f_y)+w_0(6ff_yf_{xy}\\&-3f^2f_{yyx}-6f_xf_y^2+3f_xf_{yy}f)=0,\end{aligned} \tag{5}$$

while (2) is satisfied identically under the transformation (3).

After performing some tedious but direct calculation, one can find that the trilinear equation (5) possesses the following variable separation solution

$$f=p_1(x,t)+p_2(x,t)q(y,t), \tag{6}$$

where $p_1\equiv p_1(x,t)$, $p_2\equiv p_2(x,t)$ are functions of $\{x,t\}$ and $q\equiv q(y,t)$ is a function of $\{y,t\}$. It is clear that the variables x and y now have been separated totally.

Substituting (6) into (5), we have

$$\begin{aligned}&(2f_x-f\partial_x)(p_{1xxx}p_2-p_{2xxx}p_1+3(p_{2xx}p_{1x}-p_{1xx}p_{2x})\\&+p_2p_{1t}-p_1p_{2t}+3v_0(p_1p_{2x}-p_2p_{1x}))+(p_1p_{2x}-p_2p_{1x})\\&(2p_2-q_y^{-1}f\partial_y)(-q_{3y}-q_t+3q_yw_0)=0.\end{aligned} \tag{7}$$

Because p_1 and p_2 are y in independent and q is x in independent, (7) can be divided into two equations:

$$\begin{aligned}&p_{2t}p_1-p_2p_{1t}=3v_0(p_1p_{2x}-p_2p_{1x})\\&+p_{1xxx}p_2-p_{2xxx}p_1+3(p_{2xx}p_{1x}-p_{1xx}p_{2x}),\end{aligned} \tag{8}$$

$$q_t=3q_yw_0-q_{3y}. \tag{9}$$

Thanks to the arbitrariness of the functions v_0 and w_0, the soliton solution of the NNV equation may have quite rich structures. In fact, it is not necessary to solve (8) and (9) because of the arbitrariness of the functions v_0 and w_0. In other words, if we fix the functions v_0 and w_0 as

$$\begin{aligned}v_0=&\frac{1}{3(p_1p_{2x}-p_2p_{1x})}[p_{2t}p_1-p_2p_{1t}+p_{2xxx}p_1\\&-p_{1xxx}p_2-3(p_{2xx}p_{1x}-p_{1xx}p_{2x})],\end{aligned} \tag{10}$$

$$w_0=\frac{1}{3q_y}\left(q_t+q_{3y}\right), \tag{11}$$

then p_1, p_2 and q become three arbitrary functions.

Finally, substituting (6) into (3) we find that the NNV equation possesses an exact solution:

$$u=-2(\ln f)_{xy}=-\frac{2p_{2x}q_y}{p_1+p_2q}+\frac{2(p_{1x}+p_{2x}q)p_2q_y}{(p_1+p_2q)^2}, \tag{12}$$

$$v=-2(\ln f)_{xx}+v_0=-\frac{2p_{1xx}+p_{2xx}q}{p_1+p_2q}+\frac{2(p_{1x}+p_{2x}q)^2}{(p_1+p_2q)^2}+v_0, \tag{13}$$

$$w=-2(\ln f)_{yy}+w_0=-\frac{2p_2q_{yy}}{p_1+p_2q}+\frac{2p_2^2q_y^2}{(p_1+p_2q)^2}+w_0. \tag{14}$$

From (10)–(14), we know that for general selections of p_1, p_2 and q there may be some singularities for u, v and w. We have to choose the functions p_1, p_2 and q carefully to avoid the singularities. When functions $p_1(x,t)$, $p_2(x,t)$ and $q(y,t)$ are selected to avoid the singularities of (12), (13) and (14), (12), (13) and (14) reveal quite abundant soliton structures.

Here we give only one example which describes the interaction between a line soliton and a y-periodic soliton. Taking

$$p_1 = 1 + \alpha_1 \exp(2\xi_1) + \exp(\xi_2) + \alpha_{12}\exp(2\xi_1 + \xi_2),$$
$$p_2 = -\alpha_{11}\exp(\xi_1) - \alpha\alpha_{12}\exp(\xi_1 + \xi_2),$$
$$q = \cos(\eta), \tag{15}$$

$$\xi_1 = k_1 x - \omega_1 t, \quad \xi_2 = k_2 x - \omega_2 t, \quad \eta = l_2 y,$$

$$\omega_1 = k_1^3, \quad \omega_2 = \frac{k_2\left(3k_1^4 + 3l_2^2 - k_1^2 k_2^2\right)}{2k_1^2},$$

$$\alpha_1 = \frac{k}{4k_1^4}, \quad \alpha\alpha_{12} = \frac{L}{k_1^2}, \quad \alpha_{12} = \frac{kL^2}{4k_1^4}, \quad \alpha_{11} = \frac{1}{k_1^2},$$

$$L = \frac{(k_1 - k_2)^2 - \frac{l_2^2}{k_1^2}}{(k_1 + k_2)^2 - \frac{l_2^2}{k_1^2}},$$

the resulting potential function

$$v = -2(\log f)_{xx} + v_0 \tag{16}$$

describes the interactions between a line soliton and a y-periodic soliton, where

$$f = p_1 + p_2 q = 1 + \frac{k}{4k_1^4}\exp(2\xi_1) - \frac{1}{k_1^2}\exp(\xi_1) \times \cos(l_2 y) + \exp(\xi_2) \tag{17}$$
$$\times \left[1 + \frac{kL^2}{4k_1^4}\exp(2\xi_1) - \frac{L}{k_1^2}\exp(\xi_1)\cos(l_2 y)\right],$$

$$v_0 = \frac{1}{3(p_1 p_{2x} - p_2 p_{1x})}[p_{2t}p_1 - p_2 p_{1t} + p_1 p_{2xxx} - p_2 p_{1xxx} - 3(p_{2xx}p_{1x} - p_{1xx}p_{2x})] \equiv 0. \tag{18}$$

3. Interaction between a line Soliton and a *Y*-Periodic Soliton

3.1. Analytical Representation of the Interaction

When we assume that $k_1 > 0$, $k_2 > 0$ and $\omega_2/k_2 > \omega_1/k_1$, we obtain the expressions of two separated line soliton and periodic solutions before and after interactions as follows:

$$f(\xi_1, \eta, L) = \exp(\xi_2)\left[1 + \frac{kL^2}{4k_1^4}\exp(2\xi_1) - \frac{L}{k_1^2} \times \exp(\xi_1)\cos(\eta)\right], \tag{19}$$

$$f(\xi_2) = 1 + \exp(\xi_2) \tag{20}$$

and

$$f(\xi_1, \eta) = \left[1 + \frac{k}{4k_1^4}\exp(2\xi_1) - \frac{1}{k_1^2}\exp(\xi_1)\cos(\eta)\right], \tag{21}$$

$$f(\xi_2, L) = \frac{k}{4k_1^4}\exp(2\xi_1)\left[1 + L^2\exp(\xi_2)\right], \tag{22}$$

respectively, where the subscripts 1 and 2 denote the coordinates of the y-periodic soliton and the line soliton, respectively. Taking into account that v is unchanged even if f is multiple by $\exp(ax+b)$ with a and b independent of x, we have only to consider the form of f. In a nutshell we have the following results

$$[f_1(\xi_1 + \Gamma, \eta), f_2(\xi_2)] \longrightarrow$$
$$[f_1(\xi_1, \eta), f_2(\xi_2 + 2\Gamma)] \quad \text{for } L > 0,$$
$$[f_1(\xi_1, \eta + \pi), f_2(\xi_2 + 2\Gamma)] \quad \text{for } L < 0, \tag{23}$$

where $\Gamma = \log|L|$. This expression shows that the phase shift due to the interaction is determined only by the coefficient L. The phase shift in the propagation direction is determined by the magnitude of L, while that in the transverse direction by the sign of L. The condition $L = \pm\infty$ corresponds to the phase shift in the propagation direction becoming infinite, $\Gamma \longrightarrow +\infty$. In the case $k_1 k_2 > 0$, which we treat in this paper, this relation means that the time periodic of the interaction becomes infinite. The use of this condition and the appropriate limits, i.e. $\xi_1 \longrightarrow -\infty$, $\xi_2 \longrightarrow -\infty$ and $2\xi_1 + \xi_2 + 2\log|L| \sim O\ (1)$ in (17) gives the expression

$$v_r = -\frac{(2k_1 + k_2)^2}{2}\sec h^2\left(\left(k_1 + \frac{k_2}{2}\right)x - \left(\omega_1 + \frac{\omega_2}{2}\right)t + \frac{\sigma}{2}\right), \tag{24}$$

where $\sigma = \log\left(\frac{L^2}{4k_1^4}\right)$. It means that the interaction between one line soliton and one y-periodic soliton gives rise to a new one line soliton, and the subscript r denotes resonant. The resonant line soliton, v_r, satisfies the conservation law related to the momentum

$$\int_{-\infty}^{\infty} v_r \mathrm{d}x = \int_{-\infty}^{\infty} v_1 \mathrm{d}x + \int_{-\infty}^{\infty} v_2 \mathrm{d}x = -2(2k_1 + k_2), \quad (25)$$

where v_1 and v_2 are the line soliton and the y-periodic soliton before the interaction.

The condition $L = 0$ corresponds to the phase shift in the propagation direction becoming negative infinite, $\Gamma \longrightarrow -\infty \, (k_1 k_2 > 0)$. This corresponds to long-range interaction, which means that two solitons can interact infinitely apart from each other. Taking the limits $\xi_1 \longrightarrow \infty$, $\xi_2 \longrightarrow \infty$ and $\xi_1 + \log|L| \longrightarrow -\infty$ keeping $2\xi_1 - \xi_2 \sim O(1)$ in (17), we obtain

$$v = -\frac{(2k_1 - k_2)^2}{2} \operatorname{sec}h^2 \times \left(\left(k_1 - \frac{k_2}{2}\right)x - \left(\omega_1 - \frac{\omega_2}{2}\right)t + \frac{\sigma}{2}\right), \quad (26)$$

where $\sigma = \log\left(k/4k_1^4\right)$. This is regarded as a new one line soliton.

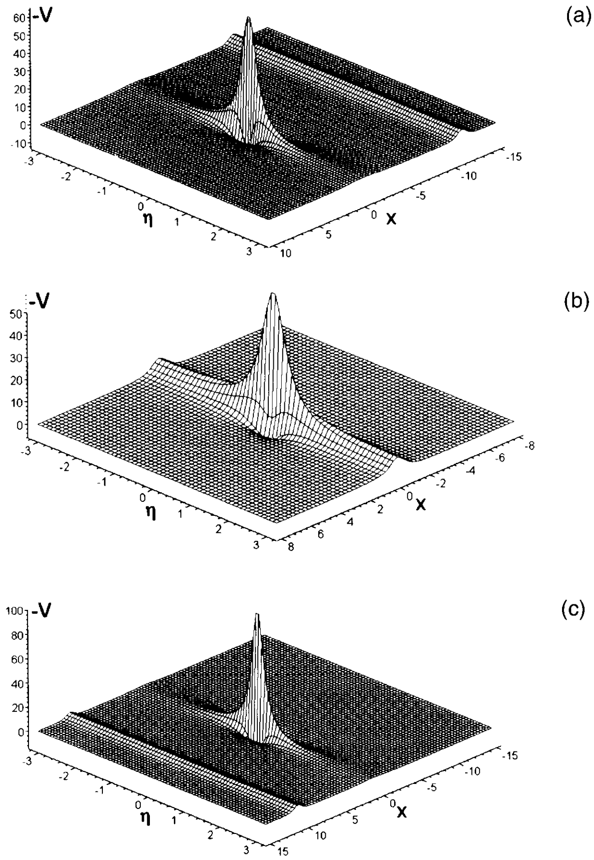

Fig. 1. The interaction plot between a line soliton and a y-periodic soliton with $(k_1, l_2, k_2) = (1, 9/\sqrt{3}, 3)$. $L = 2.09$ and $(\omega_2/k_2, \omega_1/k_1) = (75/2, 1)$. (a) $t = -0.3$, (b) $t = 0$, (c) $t = 0.3$.

3.2. Graphical Representation of Interactions

In this section we give a graphical representation of the time-evolution of the collision between a line soliton and a y-periodic soliton in several case, in which $L > 0$, $L < 0$, $\Gamma \longrightarrow +\infty$ and $L \longrightarrow 0$.

Figure 1 shows the interaction between a line soliton and a y-periodic soliton for $L > 0$, where the function f is determined by (17), and

$$\xi_1 = x - t, \quad \xi_2 = 3x - \frac{225}{2}t, \quad \eta = 3\sqrt{3}y. \quad (27)$$

In Figs. 1(a) – (c), the time is taken as -0.3, 0, 0.3, respectively. From Fig. 1, we can see that one line soliton and one y-periodic soliton before interaction are still one line soliton and one y-periodic soliton after interaction. Because $L > 0$ and L is small, the phase shift in propagation is not noticeable.

Figure 2 shows the interaction ($L = -0.2$) between a line soliton and a y-periodic soliton for $L < 0$, where function f is the same as in Fig. 1 and

$$\xi_1 = 0.5x - 0.125t, \quad \xi_2 = 1.426x - 1.937t, \quad \eta = \frac{\sqrt{3}}{3}y. \quad (28)$$

In Figs. 2(a) – (c), the time is taken as -15, 0, 15, respectively. Because $L < 0$ in Fig. 2, the occurrence of the transverse phase shift is expected.

Fig. 3 shows interaction plots of a line soliton and a y-periodic soliton for $L \longrightarrow \infty \, (L = 1.064 \times 10^5)$ $(\Gamma \longrightarrow \infty)$, where f is also determined by (17) and

$$\xi_1 = 1.60x - 4.096t, \quad \xi_2 = 1.648x - 30.156t, \quad \eta = 3\sqrt{3}y. \quad (29)$$

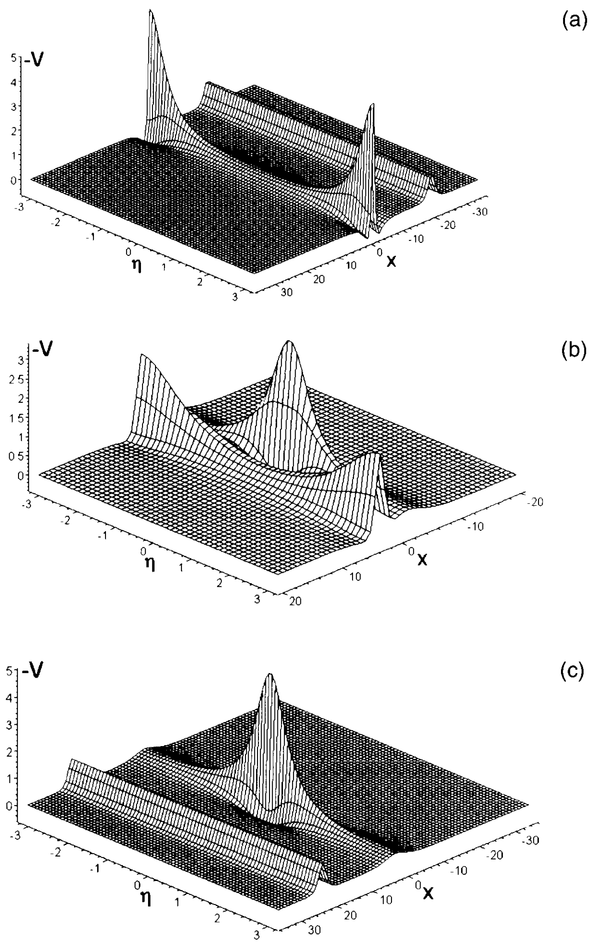

Fig. 2. The interaction plots between a line soliton and a *y*-periodic soliton with $(k_1, l_2, k_2) = (0.50, \mathrm{B}\sqrt{3}/3, 1.426)$. $L = -0.20$ and $(\omega_2/k_2, \omega_1/k_1) = (1.36, 1.94)$. (a) $t = -15$, (b) $t = 0$, (c) $t = 15$.

In Figure 3 the time is taken as -1.0, -0.8, -0.2, 0.02, 0.6, 6. The interaction is expected to yield an intermediate soliton whose propagation persists over a comparatively long time. Initially they are separated well enough to regarded as two independent solitons (Fig. 3(a)). As the *y*-periodic soliton is approaching the line soliton, the hump is decreasing and the trough is increasing which looks as if the line soliton swallowed the hump (Fig. 3(c), 3(d)). After sufficiently long time the intermediate one line soliton begins to separate two line solitons (Fig. 3(f)). This is because we have chosen parameters which do not satisfy the resonant condition strictly. If we choose parameters which satisfy the resonant condition strictly, we obtain the exact expression of the resonant line soliton.

Figure 4 are the singular interacting plots, in the plots, $L = -1.3 \times 10^{-4}$ ($\Gamma \longrightarrow -\infty$ is satisfied. This case is regarded as long-range interaction. Here we have chosen

$$\xi_1 = 0.6x - 0.216t, \quad \xi_2 = 1.562x - 1.107t, \quad (30)$$

$$\eta = \frac{\sqrt{3}}{3} y.$$

In Fig. 4, the time is taken as -20, -12, -4, 4, 12, 20. Because $L \longrightarrow 0$ and < 0, therefore the transverse phase shift is also expected. We note that the two solitons are far separated during the interaction.

4. Summary and Discussions

With the help of the variable separation approach we obtain a solution which describes the interaction between a line soliton and a *y*-periodic soliton about NNV equation conveniently. We have discussed the interaction between a line soliton and a *y*-periodic soliton both analytically and in graphically. The interaction is related to the interaction coefficient L. The magnitude of interaction coefficient is related to the phase shift in the propagation direction, and its sign is related to that in the transverse direction. The negative sign corresponds to the transverse phase with π, which is very peculiar to higher dimensional problems. We have shown that there are two types of singular cases, one is called the resonant interaction where $L \longrightarrow \infty$, and the other is called long-range interaction where $L \longrightarrow 0$. The resonant interaction is characterized by the resonant soliton that satisfies the conservation law. In the case of resonant and long-range interaction, the interaction gives rise to the simpler structure, i.e. the line soliton.

The variable separation procedure enables a simple approach to find exact solutions of a $(2+1)$-dimensional integrable model. Using this method, we can study the interaction between different types of soliton conveniently. From the variable separation form of (6) one can get a solution which describes the interaction between two dromions for a field u. Whether solutions to describe the interaction between a *y*-periodic soliton and the algebraic soliton, between inclined periodic solitons, etc., can be

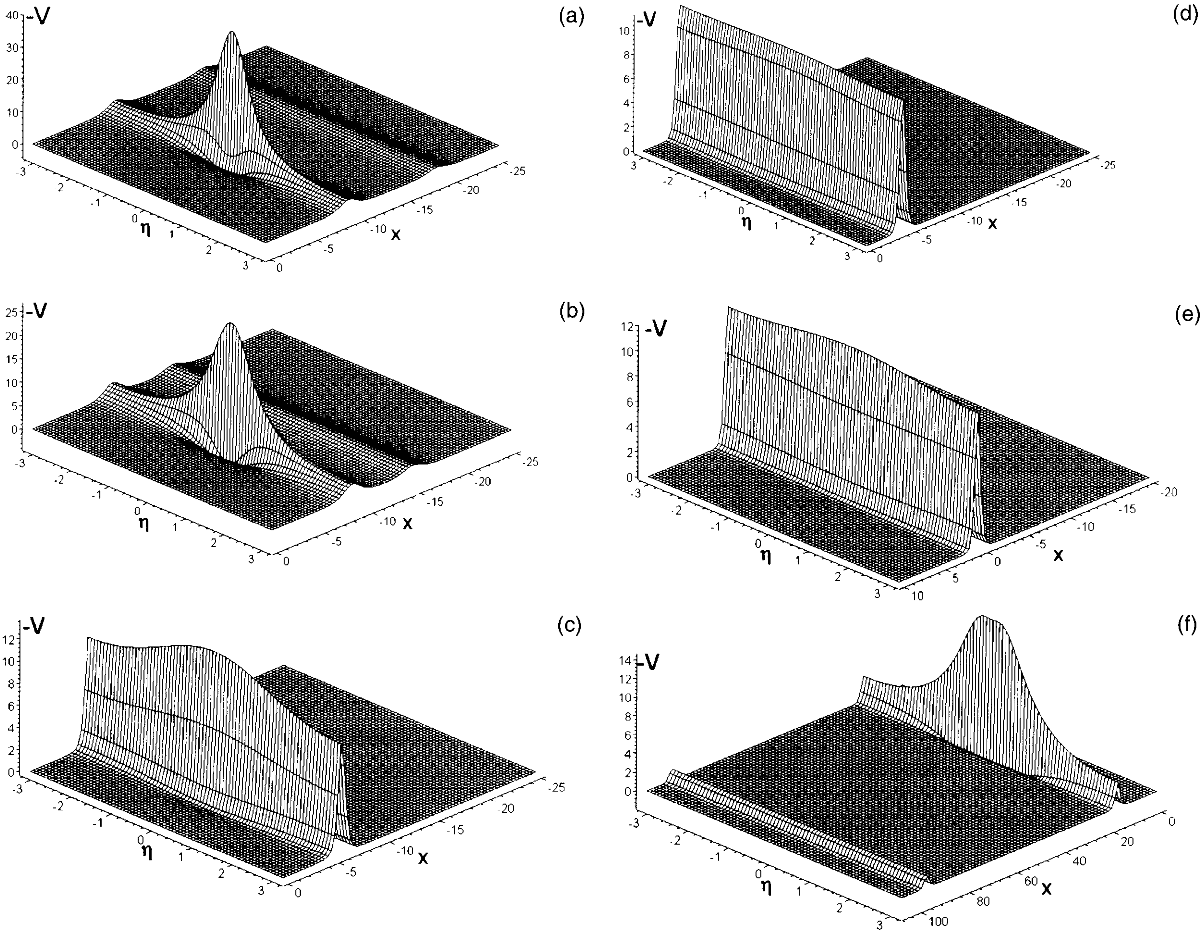

Fig. 3. The interaction plots between a line soliton and a *y*-periodic soliton with $(k_1, l_2, k_2) = (1.60, 3\sqrt{3}, 1.6476)$. $L = 1{,}06 \times 10^5$ and $(\omega_2/k_2, \omega_1/k_1) = (18.30, 2.56)$. (a) $t = -1.0$, (b) $t = -0.8$, (c) $t = -0.2$, (d) $t = 0.02$, (e) $t = 0.6$, (f) $t = 6.0$.

obtained via the variable separation approach will be further studied.

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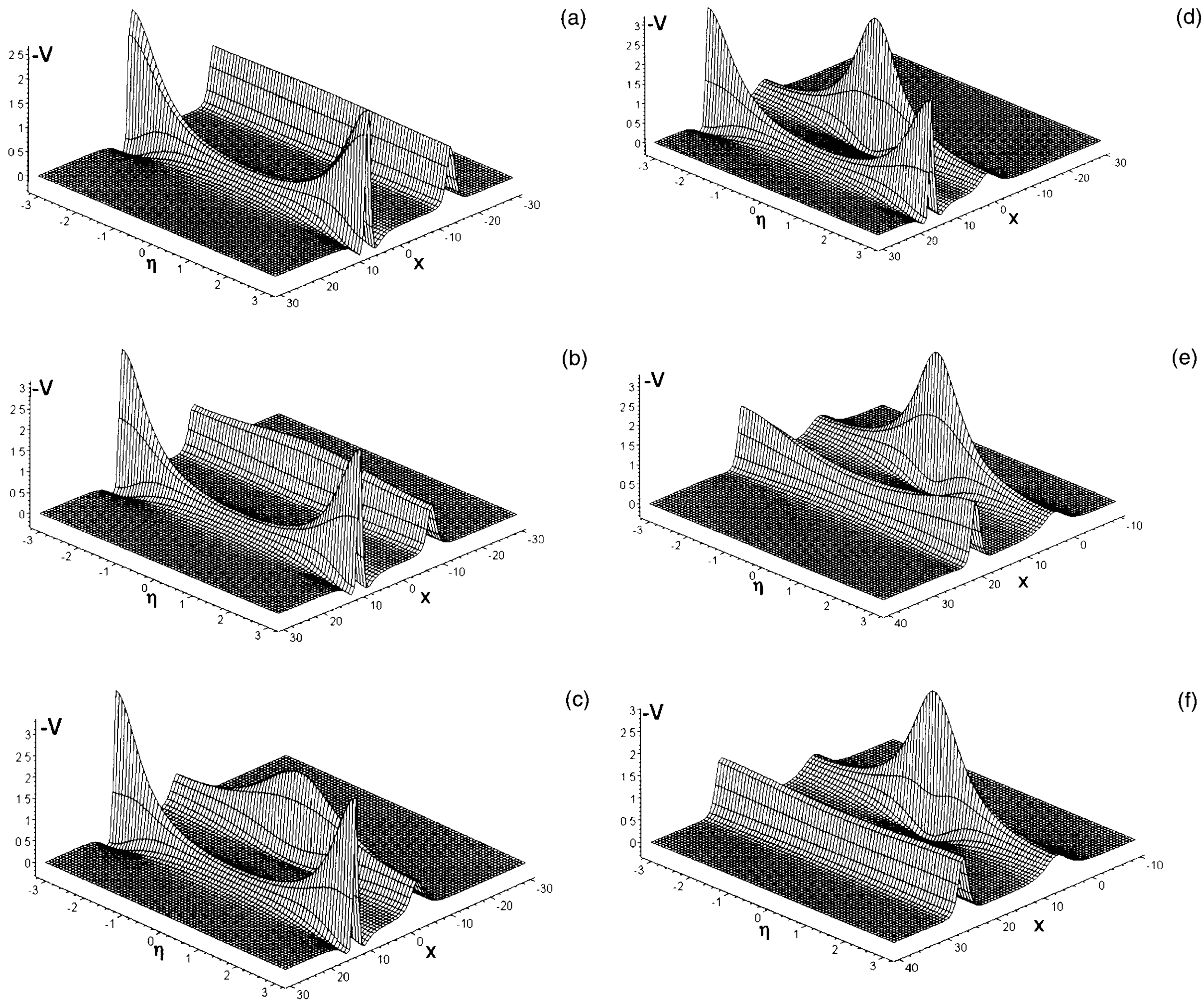

Fig. 4. The interaction plots between a line soliton and a *y*-periodic soliton with $(k_1, l_2, k_2) = (0.6, \sqrt{3}/3, 1.562)$. $L = -1.3 \times 10^{-4}$ and $(\omega_2/k_2, \omega_1/k_1) = (0.709, 0360)$. (a) $t = -20$, (b) $t = -12$, (c) $t = -4$, (d) $t = 4$, (e) $t = 12$, (f) $t = 20$.

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